

## BILIPSCHITZ MAPPINGS WITH DERIVATIVES OF BOUNDED VARIATION

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*Abstract*

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Let  $\Omega \subset \mathbb{R}^n$  be open and suppose that  $f: \Omega \rightarrow \mathbb{R}^n$  is a bilipschitz mapping such that  $Df \in BV_{\text{loc}}(\Omega, \mathbb{R}^{n^2})$ . We show that under these assumptions the inverse satisfies  $Df^{-1} \in BV_{\text{loc}}(f(\Omega), \mathbb{R}^{n^2})$ .

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### 1. Introduction

Suppose that  $\Omega \subset \mathbb{R}^n$  is an open set and let  $f: \Omega \rightarrow f(\Omega) \subset \mathbb{R}^n$  be a homeomorphism. In this paper we address the issue of the regularity of  $f^{-1}$  under regularity assumptions on  $f$ . The starting point for us is the following very recent result from [4] (see Preliminaries for the definition of the space  $BV$ ).

**Theorem 1.1.** *Let  $\Omega, \Omega' \subset \mathbb{R}^2$  be open and suppose that  $f: \Omega \rightarrow \Omega'$  is a homeomorphism. Then  $f \in BV_{\text{loc}}(\Omega; \mathbb{R}^2)$  if and only if  $f^{-1} \in BV_{\text{loc}}(\Omega'; \mathbb{R}^2)$ . Moreover, both  $f$  and  $f^{-1}$  are differentiable almost everywhere.*

In the same paper we also studied the conditions that guarantee that  $f^{-1} \in BV_{\text{loc}}(\Omega, \mathbb{R}^n)$  in higher dimensions. With some additional assumptions (namely that  $f$  is a mapping of finite distortion) it is moreover possible to prove that  $f^{-1} \in W_{\text{loc}}^{1,1}(\Omega, \mathbb{R}^n)$  (see [2], [3] and [5]).

In this paper we want to address the issue of regularity of the second derivative of  $f^{-1}$ . The classical inverse function theorem states that if  $f$  is  $C^2$  and  $J_f(x_0) \neq 0$  then there is a small neighborhood of  $x_0$  where  $f$  is homeomorphism and  $f^{-1}$  is  $C^2$ . We will assume that  $f$  is a bilipschitz mapping and show that  $Df^{-1} \in BV_{\text{loc}}$  provided that  $Df \in BV_{\text{loc}}$ . This resembles the result from [6] that the inverse of bilipschitz Delta-convex mapping is again Delta-convex.

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**Theorem 1.2.** *Let  $\Omega, \Omega' \subset \mathbb{R}^n$  be open and suppose that  $f: \Omega \rightarrow \Omega'$  is a bilipschitz mapping such that  $Df \in BV_{\text{loc}}(\Omega; \mathbb{R}^{n^2})$ . Then  $Df^{-1} \in BV_{\text{loc}}(\Omega'; \mathbb{R}^{n^2})$ .*

It is moreover possible to show that  $Df^{-1}$  belongs to the Sobolev space  $W_{\text{loc}}^{1,p}$  if  $Df \in W_{\text{loc}}^{1,p}$ .

**Theorem 1.3.** *Let  $\Omega, \Omega' \subset \mathbb{R}^n$  be open,  $p \geq 1$  and suppose that  $f: \Omega \rightarrow \Omega'$  is a bilipschitz mapping such that  $Df \in W_{\text{loc}}^{1,p}(\Omega; \mathbb{R}^{n^2})$ . Then  $Df^{-1} \in W_{\text{loc}}^{1,p}(\Omega'; \mathbb{R}^{n^2})$ .*

Let us make a comment on our assumptions. Let  $\alpha > 1$  and consider the function  $f: (-1, 1) \rightarrow \mathbb{R}$  defined as  $f(x) = |x|^\alpha \operatorname{sgn} x$ . Then it is easy to check that  $f$  is Lipschitz, homeomorphism,  $Df \in W^{1,1}((-1, 1))$ , but  $Df^{-1} \notin BV_{\text{loc}}((-1, 1))$ . Thus the assumption that  $f^{-1}$  is Lipschitz cannot be omitted.

In Section 4 we give an example which shows that Theorem 1.2 is not valid in dimension  $n \geq 4$  without the assumption that  $f$  is Lipschitz. If  $n = 1$ , then  $Df \in BV$  implies that  $Df$  is bounded and that  $f$  is Lipschitz, and thus this assumption is redundant. We would like to know if a homeomorphism  $f: \Omega \rightarrow \Omega'$  such that  $f^{-1}$  is Lipschitz and  $Df \in BV_{\text{loc}}(\Omega; \mathbb{R}^{n^2})$  must satisfy  $Df^{-1} \in BV_{\text{loc}}(\Omega'; \mathbb{R}^{n^2})$  in dimensions  $n = 2$  and  $n = 3$ . Unfortunately our method of the proof and our counterexample do not provide an answer to this question.

## 2. Preliminaries

By  $\mathbf{e}_1, \dots, \mathbf{e}_n$  we denote the canonical basis in  $\mathbb{R}^n$ . For  $x \in \mathbb{R}^n$  we write  $x_1, \dots, x_n$  for its coordinates, i.e.  $x = \sum_{i=1}^n x_i \mathbf{e}_i$ . The euclidean distance of  $x, y \in \mathbb{R}^n$  is denoted by  $|x - y|$  and the norm of the  $n$  times  $n$  matrix  $A$  is denoted by  $\|A\|$ .

In the whole paper  $\Omega$  will denote an open subset of  $\mathbb{R}^n$ . We say that  $F: \Omega \rightarrow \mathbb{R}^n$  is a Lipschitz map if there is a constant  $K > 0$  such that

$$|F(x) - F(y)| \leq K|x - y|$$

for every  $x, y \in \Omega$ . Further  $F$  is said to be bilipschitz if it is an invertible mapping and both  $F: \Omega \rightarrow \mathbb{R}^n$  and  $F^{-1}: F(\Omega) \rightarrow \mathbb{R}^n$  are Lipschitz.

The Lebesgue measure of a set  $A \subset \mathbb{R}^n$  is denoted by  $\mathcal{L}_n(A)$ . A mapping  $f: \Omega \rightarrow \mathbb{R}^n$  is said to satisfy the Lusin condition (N) if  $\mathcal{L}_n(f(A)) = 0$  for every  $A \subset \Omega$  such that  $\mathcal{L}_n(A) = 0$ .

Let  $\Omega \subset \mathbb{R}^n$  be open and  $m \in \mathbb{N}$ . A function  $h \in L^1(\Omega)$  is of bounded variation,  $h \in BV(\Omega)$ , if the distributional partial derivatives of  $h$  are

measures with finite total variation in  $\Omega$ : there are Radon (signed) measures  $\mu_1, \dots, \mu_n$  defined in  $\Omega$  so that for  $i = 1, \dots, n$ ,  $|\mu_i|(\Omega) < \infty$  and

$$\int_{\Omega} h D_i \varphi \, dx = - \int_{\Omega} \varphi \, d\mu_i$$

for all  $\varphi \in C_0^\infty(\Omega)$ . We say that  $f \in L^1(\Omega, \mathbb{R}^m)$  belongs to  $BV(\Omega, \mathbb{R}^m)$  if the coordinate functions of  $f$  belong to  $BV(\Omega)$ . Analogously we define the Sobolev space:  $f \in W^{1,1}(\Omega, \mathbb{R}^m)$  if  $f \in L^1(\Omega, \mathbb{R}^m)$  and the distributional derivatives of the coordinate functions are in  $L^1(\Omega, \mathbb{R}^n)$ . Further,  $f \in BV_{\text{loc}}(\Omega, \mathbb{R}^m)$  (or  $f \in W_{\text{loc}}^{1,1}(\Omega, \mathbb{R}^m)$ ) requires that  $f \in BV(\Omega', \mathbb{R}^m)$  (or  $f \in W^{1,1}(\Omega', \mathbb{R}^m)$ ) for each open  $\Omega' \Subset \Omega$ . For an introduction to the theory of  $BV$  and  $W^{1,1}$  spaces see [1], [7]. The function  $h: \Omega \rightarrow \mathbb{R}^m$  is said to be a representative of  $g: \Omega \rightarrow \mathbb{R}^m$  if  $h = g$  almost everywhere with respect to Lebesgue measure.

For function  $f: (a, b) \rightarrow \mathbb{R}^m$  we define

$$V(f, (a, b)) := \sup \left\{ \sum_{i=1}^k |f(a_i) - f(b_i)| : (a_i, b_i) \text{ are pairwise disjoint intervals in } (a, b) \right\}.$$

The function  $f$  is said to have finite variation if  $V(f, (a, b)) < \infty$ .

It is a well-known fact (see e.g. [1, Section 3.11]) that a mapping  $u \in L_{\text{loc}}^1(\Omega, \mathbb{R}^m)$  is in  $BV_{\text{loc}}(\Omega, \mathbb{R}^m)$  (or in  $W_{\text{loc}}^{1,1}(\Omega, \mathbb{R}^m)$ ) if and only if there is a representative which has bounded variation (or is an absolutely continuous function) on almost all lines parallel to coordinate axes and the variation on these lines is integrable. More precisely, let  $i \in \{1, 2, \dots, n\}$  and denote by  $\pi_i$  the projection on the hyperplane perpendicular to the  $x_i$ -axis. Suppose that  $Q(c, r) := (c_1 - r, c_1 + r) \times \dots \times (c_n - r, c_n + r) \subset \Omega$  for some  $c \in \mathbb{R}^n$ ,  $r > 0$  and set  $Q^i(c, r) = \pi_i(Q(c, r))$ . Let  $y \in Q^i(c, r)$  and denote

$$u_{i,y}(t) = u(y + t\mathbf{e}_i) \text{ for } t \in (c_i - r, c_i + r).$$

**Theorem 2.1.** *Let  $\Omega \subset \mathbb{R}^n$  be open and let  $u \in L_{\text{loc}}^1(\Omega, \mathbb{R}^m)$ .*

- (i) *Then  $u \in W_{\text{loc}}^{1,1}(\Omega, \mathbb{R}^m)$  if and only if the following happens. For every cube  $Q(c, r) \Subset \Omega$  and for every  $i \in \{1, \dots, n\}$  there is a representative  $\tilde{u}$  of  $u$  such that the function  $\tilde{u}_{i,y}(t)$  is absolutely continuous on  $(c_i - r, c_i + r)$  (i.e. each coordinate function is absolutely*

continuous) for  $\mathcal{L}_{n-1}$  almost every  $y \in Q^i(c, r)$  and moreover

$$(2.1) \quad \int_{Q^i(c, r)} \int_{c_i - r}^{c_i + r} |\nabla \tilde{u}_{i, y}(t)| dt dy < \infty.$$

- (ii) Then  $u \in BV_{\text{loc}}(\Omega, \mathbb{R}^m)$  if and only if the following happens. For every cube  $Q(c, r) \Subset \Omega$  and for every  $i \in \{1, \dots, n\}$  there is a representative  $\tilde{u}$  of  $u$  such that the function  $\tilde{u}_{i, y}(t)$  has bounded variation on  $(c_i - r, c_i + r)$  for  $\mathcal{L}_{n-1}$  almost every  $y \in Q^i(c, r)$  and moreover

$$(2.2) \quad \int_{Q^i(c, r)} V(\tilde{u}_{i, y}, (c_i - r, c_i + r)) dy < \infty.$$

We shall also need that the composition of  $BV$  function and a homeomorphism with Lipschitz inverse is in  $BV$  (see [1, Theorem 3.16 and Corollary 3.19]).

**Theorem 2.2.** *Let  $\Omega, \Omega' \subset \mathbb{R}^n$  be open and let  $u: \Omega \rightarrow \mathbb{R}^m$ . Suppose that  $F: \Omega \rightarrow \Omega'$  is Lipschitz and homeomorphism.*

- (i) *If  $u \in BV_{\text{loc}}(\Omega, \mathbb{R}^m)$ , then  $u \circ F^{-1} \in BV(\Omega', \mathbb{R}^m)$ .*  
(ii) *If  $u \in W_{\text{loc}}^{1,1}(\Omega, \mathbb{R}^m)$  and  $F^{-1}$  is Lipschitz, then  $u \circ F^{-1} \in W_{\text{loc}}^{1,1}(\Omega', \mathbb{R}^m)$  and*

$$Du \circ F^{-1}(y) = Du(F^{-1}(y))DF^{-1}(y) \text{ for almost every } y \in \Omega'.$$

### 3. Regularity of the inverse

*Proof of Theorem 1.2:* We want to show that  $Df^{-1}$  has bounded variation on almost all lines parallel to coordinate axes and therefore  $Df^{-1} \in BV_{\text{loc}}$  (see Theorem 2.1 (ii)). Fix  $Q(c, r) \Subset f(\Omega)$  and  $i \in \{1, \dots, n\}$ .

From Theorem 2.2 we know that  $Df \circ f^{-1} \in BV_{\text{loc}}$ . Denote by  $h$  a good representative of  $Df \circ f^{-1}$  from Theorem 2.1 (ii) and set  $h_{i, y}(t) := h(y + te_i)$  for  $y \in Q^i(c, r)$ . From (2.2) we have

$$(3.1) \quad \int_{Q^i(c, r)} V(h_{i, y}, (c_i - r, c_i + r)) dy < \infty.$$

Denote

$$A = \{x \in Q(c, r) : h(x) = Df \circ f^{-1}(x), f^{-1} \text{ is differentiable at } x \\ \text{and } f \text{ is differentiable at } f^{-1}(x)\}.$$

Lipschitz functions are differentiable almost everywhere and map Lebesgue null sets to Lebesgue null sets and therefore  $\mathcal{L}_n(A) = \mathcal{L}_n(Q(c, r))$ . From the definition of  $A$  we have

$$(3.2) \quad Df^{-1}(x)Df(f^{-1}(x)) = I \text{ for every } x \in A.$$

Fix  $y \in Q^i(c, r)$  and let  $\{(a_j, b_j)\}_{j=1}^k$  be a system of pairwise disjoint subintervals of  $(c_i - r, c_i + r)$  such that  $A_j := y + a_j \mathbf{e}_i \in A$  and  $B_j := y + b_j \mathbf{e}_i \in A$  for every  $j$ . Plainly  $\|Df^{-1}(x)\| \leq K$  where  $K$  denotes the Lipschitz constant of  $f^{-1}$ . Together with (3.2) this imply

$$\begin{aligned}
 & \sum_{j=1}^k \|Df^{-1}(A_j) - Df^{-1}(B_j)\| \\
 (3.3) \quad &= \sum_{j=1}^k \|Df^{-1}(A_j)(Df(f^{-1}(B_j)) - Df(f^{-1}(A_j)))Df^{-1}(B_j)\| \\
 &\leq K^2 \sum_{j=1}^k \|Df(f^{-1}(B_j)) - Df(f^{-1}(A_j))\| \\
 &\leq CV(h_{i,y}, (c_i - r, c_i + r)).
 \end{aligned}$$

From (3.1) and  $\mathcal{L}_n(A) = \mathcal{L}_n(Q(c, r))$  we know that  $V(h_{i,y}, (c_i - r, c_i + r)) < \infty$  and  $\mathcal{L}_1(\pi_i^{-1}(y) \cap A) = 2r$  for  $\mathcal{L}_{n-1}$  almost every  $y$ . Fix such a  $y \in Q^i(c, r)$ . From (3.3) and elementary properties of functions of bounded variation we obtain that there is a function  $\tilde{u}_{i,y}: (c_i - r, c_i + r) \rightarrow \mathbb{R}^{n^2}$  such that  $Df^{-1}(y + t\mathbf{e}_i) = \tilde{u}_{i,y}(t)$  for every  $t \in (c_i - r, c_i + r) \cap A$  and

$$(3.4) \quad V(\tilde{u}_{i,y}, (c_i - r, c_i + r)) \leq CV(h_{i,y}, (c_i - r, c_i + r)).$$

It follows that there is a function  $\tilde{u}$  such that  $\tilde{u}(x) = Df^{-1}(x)$  almost everywhere and this new representative has bounded variation on  $Q(c, r) \cap \pi_i^{-1}(y)$  for  $\mathcal{L}_{n-1}$  almost every  $y \in Q^i(c, r)$ . Now (3.4) and (3.1) yields

$$(3.5) \quad \int_{Q^i(c, r)} V(\tilde{u}_{i,y}, (c_i - r, c_i + r)) dy < \infty$$

which verifies (2.2) for  $Df^{-1}$ .  $\square$

*Proof of Theorem 1.3:* First let us prove the theorem in the case  $p = 1$ . The proof of this case is analogous to the previous proof and therefore we only sketch it and point out the differences. From Theorem 2.2 (ii) we know that  $Df \circ f^{-1} \in W_{\text{loc}}^{1,1}$ . Fix  $y \in Q^i(c, r)$  such that  $h_{i,y}$  is absolutely continuous on  $(c_i - r, c_i + r)$ . Given  $\varepsilon > 0$  find  $\delta > 0$  from the absolute continuity of  $h_{i,y}$ . Choose  $A_j$  and  $B_j$  as before and moreover assume

that  $\sum_{j=1}^k |A_j - B_j| < \delta$ . Analogously to (3.3) we obtain

$$\sum_{j=1}^k \|Df^{-1}(A_j) - Df^{-1}(B_j)\| < C\varepsilon.$$

Reasoning analogously to the previous proof we conclude that  $Df^{-1}$  has a representative which is absolutely continuous on almost all lines parallel to coordinate axes. On those lines we have

$$V(\tilde{u}_{i,y}, (c_i - r, c_i + r)) = \int_{c_i - r}^{c_i + r} |\nabla \tilde{u}_{i,y}(t)| dt.$$

From Theorem 1.2 and Theorem 2.1 (ii) we already know (2.2) and thus we obtain (2.1).

Now let us return to the case  $p > 1$ . We already know that  $Df^{-1} \in W_{\text{loc}}^{1,1}$ . Therefore we can use Theorem 2.2 (ii) and differentiate twice the identity  $f \circ f^{-1}(y) = y$  to obtain

$$(3.6) \quad D^2 f(f^{-1}(y)) Df^{-1}(y) Df^{-1}(y) + Df(f^{-1}(y)) D^2 f^{-1}(y) = 0.$$

Here and in the sequel we identify the second derivative with an linear operator from  $\mathbb{R}^{n^2}$  to  $\mathbb{R}^{n^2}$ . Clearly

$$\|(Df(f^{-1}(y)))^{-1}\| \leq C, \|Df^{-1}(y)\| \leq C \text{ and } |J_{f^{-1}}(y)| \geq C$$

at almost every point since  $f$  is bilipschitz. From (3.6) and substitution formula we now obtain

$$\begin{aligned} \int_A \|D^2 f^{-1}(y)\|^p dy &\leq C \int_A \|D^2 f(f^{-1}(y))\|^p |J_{f^{-1}}(y)| dy \\ &= C \int_{f^{-1}(A)} \|D^2 f(x)\|^p dx \end{aligned}$$

for every open set  $A \Subset f(\Omega)$  and the claim follows.  $\square$

#### 4. Necessity of the Lipschitz condition for $f$ for $n \geq 4$

**Example 4.1.** Let  $n \geq 4$ . There is a homeomorphism  $f: (-1, 1)^n \rightarrow \mathbb{R}^n$  such that  $Df \in W^{1,1}((-1, 1)^n, \mathbb{R}^{n^2})$  and  $f^{-1}$  is Lipschitz, but  $Df^{-1} \notin BV_{\text{loc}}(f((-1, 1)^n), \mathbb{R}^{n^2})$ .

*Proof:* Given  $x \in \mathbb{R}^n$  we denote  $\tilde{x} = [x_1, \dots, x_{n-1}] \in \mathbb{R}^{n-1}$  and  $\|\tilde{x}\| = \sqrt{x_1^2 + \dots + x_{n-1}^2}$ .

Let  $\alpha = \frac{1}{2n}$ ,  $\beta = \frac{3}{4}$  and set

$$f(x) = \sum_{i=1}^{n-1} \mathbf{e}_i x_i \|\tilde{x}\|^{\alpha-1} + \mathbf{e}_n (x_n + \|\tilde{x}\| \sin(\|\tilde{x}\|^{-\beta}))$$

if  $\|\tilde{x}\| > 0$  and  $f(x) = \mathbf{e}_n x_n$  if  $\|\tilde{x}\| = 0$ . Our mapping  $f$  is clearly continuous and it is easy to check that  $f$  is a one-to-one map since

$$\begin{aligned} x_i \|\tilde{x}\|^{\alpha-1} &= z_i \|\tilde{z}\|^{\alpha-1} \text{ for every } i \in \{1, \dots, n-1\} \Rightarrow \\ &\Rightarrow x_i = z_i \text{ for every } i \in \{1, \dots, n-1\}. \end{aligned}$$

Therefore  $f$  is a homeomorphism.

A direct computation shows that the second partial derivatives of  $f_i$ ,  $i \in \{1, \dots, n-1\}$ , are smaller than  $C\|\tilde{x}\|^{\alpha-2}$  and therefore integrable. Moreover,

$$\frac{\partial f_n(x)}{\partial x_1} = x_1 \|\tilde{x}\|^{-1} \sin(\|\tilde{x}\|^{-\beta}) - \|\tilde{x}\| \beta \frac{x_1}{\|\tilde{x}\|^{\beta+2}} \cos(\|\tilde{x}\|^{-\beta}).$$

It is not difficult to compute that we can bound each partial derivative of this expression by  $C\|\tilde{x}\|^{1-2(\beta+1)}$ . Clearly  $2(\beta+1)-1 < n-1$  and therefore these second partial derivatives are integrable. Analogously we can estimate other second partial derivatives of  $f_n$ . Since the second derivatives of  $f$  are smooth outside the segment  $\{[0, \dots, 0, t] : t \in (-1, 1)\}$  and  $|D^2 f| \in L^1((-1, 1)^n)$  it is easy to see that  $Df \in W^{1,1}((-1, 1)^n, \mathbb{R}^{n^2})$ .

The inverse of  $f$  is given by

$$f^{-1}(y) = \sum_{i=1}^{n-1} \mathbf{e}_i y_i \|\tilde{y}\|^{\frac{1}{\alpha}-1} + \mathbf{e}_n (y_n - \|\tilde{y}\|^{\frac{1}{\alpha}} \sin(\|\tilde{y}\|^{-\frac{\beta}{\alpha}}))$$

if  $\|\tilde{y}\| > 0$  and  $f^{-1}(y) = \mathbf{e}_n y_n$  if  $\|\tilde{y}\| = 0$ . The derivative of the function  $\phi(t) = t^{\frac{1}{\alpha}} \sin(t^{-\frac{\beta}{\alpha}})$  is bounded on  $(-1, 1)$  and therefore  $\phi$  is Lipschitz. Thus it is not difficult to see that  $f^{-1}$  is Lipschitz.

The second derivative of  $f^{-1}$  is clearly continuous outside the segment  $\{[0, \dots, 0, t] : t \in \mathbb{R}\}$ . Elementary computation gives us

$$\frac{\partial (f^{-1})_n(y)}{\partial y_1} = -\frac{y_1}{\alpha} \|\tilde{y}\|^{\frac{1}{\alpha}-2} \sin(\|\tilde{y}\|^{-\frac{\beta}{\alpha}}) + \|\tilde{y}\|^{\frac{1}{\alpha}} \frac{\beta}{\alpha} \frac{y_1}{\|\tilde{y}\|^{\frac{\beta}{\alpha}+2}} \cos(\|\tilde{y}\|^{-\frac{\beta}{\alpha}})$$

and therefore the second derivative  $\frac{\partial^2 (f^{-1})_n(y)}{\partial y_1^2}$  contains some integrable terms and

$$(4.1) \quad -\|\tilde{y}\|^{\frac{1}{\alpha}} \frac{\beta^2}{\alpha^2} \frac{y_1^2}{\|\tilde{y}\|^{2\frac{\beta}{\alpha}+4}} \sin(\|\tilde{y}\|^{-\frac{\beta}{\alpha}}).$$

Since

$$2\left(\frac{\beta}{\alpha} + 1\right) - \frac{1}{\alpha} > n - 1$$

we obtain that the integral of the absolute value of (4.1) over the set

$$S = \{y \in f((-1, 1)^n) : y_1 > \frac{1}{2}\|\tilde{y}\|, \sin(\|\tilde{y}\|^{-\frac{\beta}{\alpha}}) > \frac{1}{2}\}$$

is infinite. Hence  $|D^2 f^{-1}| \notin L^1_{\text{loc}}$  and it is not difficult to deduce that  $Df^{-1} \notin BV_{\text{loc}}$ .  $\square$

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